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Gilbert Laporte, Juan A. Mesa, Federico Perea
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Abstract

This paper deals with the problem of designing a rapid transit network that reacts as well as possible to link failures. It is assumed that when a link fails, another path or another transportation mode is provided to transport passengers between the endpoints of the affected link. The goal is to build a network that attracts as many passengers as possible when a failure occurs. The problem is posed as a two-person zero-sum non-cooperative game with perfect information. Saddle points in the arising mixed enlarged game correspond to robust network designs.

Keywords: Robust Network Design, Game Theory, Saddle Points, Nash Equilibrium.

1 Introduction

The design or extension of Rapid Transit Networks (RTN) is a crucial issue in many cities. The reduction of traffic congestion, travel times, energy consumption and pollution justifies investment in transportation networks. This interest is reflected in many of research papers, see for instance Gendreau et al. (1995). An important issue in network design is the competition between the RTN to be constructed and other operating transportation modes. Indeed, when designing a new transportation system, there usually is another system already operating in the area. For instance consider the case of a metro system to be built in a city where there is already a bus network. One would like the metro to be competitive with the bus network in terms of travel time, price, comfort, etc. This situation has already been addressed in Laporte et al. (2005), where an integer linear programming problem is used to design a transportation network in the presence of a competing mode.

*Canada Research Chair in Distribution Mangement. HEC Montreal, Canada (gilbert@crt.umontreal.ca)

†Departamento de Matemática Aplicada II, University of Seville, Spain (jmesa@us.es).

‡Departamento de Matemática Aplicada II, University of Seville, Spain (perea@us.es).

Robustness is another important consideration in the design of a transportation system, which can be addressed from a variety of angles. Considering uncertainty in the input data yields a classic robustness problem. The network should be such that it works efficiently when, for instance, the origin-destination matrix is uncertain, which is a realistic situation. In Malcolm and Zenios (1994) and Mulvey et al. (1995) different choices for uncertainty parameters are treated. Berstsimas and Sim (2003) and (2004) propose models that control the degree of conservatism, therefore avoiding the classic "worst case scenario" of previous models.

A network can be also considered robust when it reacts as well as possible to operational disruptions. These can be failures due to drops in electrical power, weather conditions, mechanical breaks, human errors, etc. In Laporte et al. (2008) this problem is modelled and solved by considering that a network is robust when passengers have more than one route to reach their destinations. This way, failures on a single link will not have a drastic impact on its functioning.

In this paper we wish to design RTNs that are in competition with other transportation networks, and that will react well to failures on a single link. Our goal is to ensure that the network will be capable of serving the highest possible number of passengers in case of failure. One way of doing this is to consider the worst case scenario, that is, to maximize the utility of the network when the link that most reduces the utility fails. The solution to this problem is called a *maximin network*, which is modelled in this paper as a two-person zero-sum game, where the first player is the operator and the second is a demon who wants to attack one of its links. The payoff matrix of this game is the utility of the network built by player I when the link chosen by player II fails. Saddle points of the associated enlarged game yield networks that have a better expected utility than the maximin network.

The relationship between game theory and transportation has been investigated by a number of researchers (see Hollander and Prashker (2006) for a review of games describing transport problems). Bell (2000) has proposed a non-cooperative two-person zero-sum game between a user and an evil entity, in which the user wants to minimize his travel cost by choosing one of his available paths, and the evil entity wants to maximize such costs by changing the link performance scenarios. In an earlier reference, Fisk (1984) makes use of Nash and Stackelberg equilibria to draw a correspondence between game theory and some problems in transportation systems modelling, showing that game theory is a potential source to model and solve transportation problems.

Game theory and networks have also been combined in the network design literature. A classic example is the minimum cost spanning tree game introduced by Claus and Kleitman (1973). Kontogiannis and Zaroliagis (2008) investigate robust line planning via competitive models between line operators. Schöbel and Schwarze (2006) also present a game-theoretic model for a line planning problem and compute equilibria between different lines. Another example of the use of games in network planning is that of facility location games, see Puerto et al. (2001) and the references therein. In this paper we apply game theory to the design of a robust transportation network.

The remainder of the paper is structured as follows. In Section 2, an integer linear programming (ILP) model to design networks maximizing the minimum utility in the presence of a single link failure is presented. Section 3 is devoted to the presentation of our robust transportation design problem as a game, and the necessary concepts on game theory in order to make this paper self-contained are introduced. In Section 4 we present two examples to illustrate the concepts presented in the previous sections. The paper closes with some conclusions pointing at further research.

2 Robust Network Design as an Integer Linear Program

In this section we model the robust version of the RTN problem. The standard RTN model (see the Appendix) yields a network that optimally works when no failures occur. Here we assume that certain links may fail. In the event of arc disruption, an alternative transportation mode is usually provided to passengers, e.g., a bus service, which generates a cost for the network operator and extra travel time for passengers.

We now formulate our Robust Network Design (RND) problem. Let $K(r)$ be the utility function of network $r \in R$, where R is the set of all feasible networks that can be built. The utility can be the trip coverage, the total travel time, etc., and is adversely affected by link failures. Let $K(r, a)$ be the utility function of network $r \in R$ when link $a \in A'$ fails, A' being the set of possible links of the network. In order to build a robust network, we will seek a configuration that maximizes the utility function under the worst case scenario, that is, when the link that affects the largest number of passengers fails (note that depending on the utility function, the objective may change to that of minimizing the utility function under the worst case scenario, if we take for instance the total travel time as the utility function). Therefore, our problem is

$$\max_{r \in R} \min_{a \in A'} K(r, a) \text{ or } \min_{r \in R} \max_{a \in A'} K(r, a). \tag{1}$$

For the sake of brevity, from now on we will consider objectives to be maximized (for instance trip coverage, for which the maximin problem is to be solved). Definition 1 summarizes the concepts just introduced.

Definition 1 *Let R be a set of feasible Rapid Transit Networks, and A' be the set of possible links between stations. Let $K(r, a)$ be a certain utility of network $r \in R$ when the link $a \in A'$ is inoperative. We say that $r^* \in R$ is an optimal robust network in the maximin sense if there exists $a^* \in A'$ such that $\max_{r \in R} \min_{a \in A'} K(r, a) = K(r^*, a^*)$.*

The utility function maximized in our robustness model is trip coverage. Note that models for other utility functions are analogous. In general, the problem defined by (1) can be expressed as

$$\begin{array}{rl} \text{maximize} & b \\ \text{subject to:} & K(r, a) \geq b, \quad a \in A' \\ & r \in R. \end{array} \tag{2}$$

We now present a general ILP model yielding the network that maximizes the minimum trip coverage in case of disruption, that is, the optimal robust network in the maximin sense. The output of the model is an RTN consisting of $|L|$ lines. We now define the necessary input data, variables and constraints needed in order to express problem (2) as an ILP model.

2.1 Data and notation

1. A set $N = \{n_i;\ i = 1, 2, \ldots, I\}$ of potential sites for stations is given.
2. A set A' of feasible (bidirectional) arcs linking the elements in N is known. Therefore, we consider a directed graph $\mathcal{G}' = (N, A')$, from which arcs are to be selected to form rapid transit lines. Furthermore, there exists a graph $\mathcal{G}'' = (N, A'')$, representing the network used by the complementary mode (e.g. the street network). Let $G = (N, A)$, where $A = A' \cup A''$, be the whole network. Denote by $N(i) = \{n_j : \exists\ a \in A',\ a = (n_i, n_j)\}$ the set of adjacent nodes to n_i.
3. Every feasible arc $a = (n_i, n_j) \in A'$ has an associated length d_{ij}. The lengths of the arcs in A' usually correspond to approximate Euclidean distances if the system to be designed is underground and street network distance if it is at grade. However, forbidden regions or streets will increase the distances, and d_{ij} can also be interpreted as the generalized cost of traversing arc $a = (n_i, n_j) \in A'$.
4. For each node n_i and each arc $a \in A'$ there exists an associated cost of constructing the corresponding infrastructure: c_i is the cost of building a station at node n_i, c_a being the cost of building the link a. A bound C_{max} on the available budget is also given.
5. The mobility pattern is given by a matrix $G = (g_w) :\ w \in W$, where W is the ordered index pair set: $W = \{w = (i, j);\ n_i, n_j \in N\}$, also referred to as the set of demands.
6. The generalized cost of satisfying each demand by the complementary mode is u_w^{COM}, which depends on its (street) network.

For the model to be meaningful we need to impose that both the lengths d_{ij} on A' and the complementary mode cost u_w^{COM} on A'' must satisfy the triangle inequality.

2.2 Variables

Our ILP model includes the following variables:

1. $y_i^l = 1$, if node n_i is a station of line l; 0 otherwise.
2. $x_{ij}^l = 1$, if the arc $a = (n_i, n_j) \in A'$ belongs to line $l \in L$; 0 otherwise.
3. $x_{ij} = 1$, if the arc $a = (n_i, n_j) \in A'$ belongs to the rapid transit network; 0 otherwise.

4. $f_{ij}^{w}(s,t)$ denotes the proportion of the demand $w \in W$ going through arc $(n_i, n_j) \in A^{'}$ (the RTN) if arc $(n_s, n_t) \in A^{'}$ fails.

5. $\varphi_{ij}^{w}(s,t)$ denotes the proportion of the demand $w \in W$ going through arc $(n_i, n_j) \in A^{''}$ (the complementary mode) if arc $(n_s, n_t) \in A^{'}$ fails.

6. $p_w(s,t) = 1$ if demand w is allocated to the RTN even though arc $(n_s, n_t) \in A^{'}$ fails.

7. $u_w(s,t)$ is the generalized cost of satisfying the demand w if arc (n_s, n_t) fails. Note that the demand w will use the RTN or the complementary mode depending on which one is faster.

8. $h_l = 1$, if line l is included; 0 otherwise.

2.3 The model: objective function and constraints

With these variables the problem defined by (2) can be formulated as an ILP problem as follows:

$$\begin{array}{rll} \text{maximize} & b & \\ \text{subject to:} & \sum_{w \in W} g_w p_w(s,t) \geq b & (s,t) \in A^{'} \\ & \text{budget constraints} & \\ & \text{alignment location constraints} & \\ & \text{routing demand conservation constraints} & \\ & \text{location-allocation constraints} & \\ & \text{splitting demand constraints} & \\ & \text{binary conditions.} & \end{array} \tag{3}$$

Note that $\sum_{w \in W} g_w p_w(s,t)$ is the trip coverage of the network when arc (s,t) fails. We now describe in detail each constraint group. Note that in constraints (15) to (24) it is assumed that $(n_s, n_t) \in A'$ is the arc that is to fail.

- Budget constraints

$$\sum_{(n_i,n_j) \in A', i<j} c_{ij}\, x_{ij} + \sum_{l \in L} \sum_{n_i \in N} c_i y_i^l \leq C_{\max} \tag{4}$$

- Alignment location constraints

$$x_{ij}^l \leq y_i^l, \ (n_i, n_j) \in A', \ i < j, \ l \in L \tag{5}$$

$$x_{ij}^l \leq y_j^l, \ (n_i, n_j) \in A', \ i < j, \ l \in L \tag{6}$$

$$x_{ij}^l = x_{ji}^l, \ (n_i, n_j) \in A', \ i < j, \ l \in L \tag{7}$$

$$x_{ij}^l \leq x_{ij}, \ (n_i, n_j) \in A', \ i < j, \ l \in L \tag{8}$$

$$x_{ij} \leq \sum_{l \in L} x_{ij}^l, \ (n_i, n_j) \in A', \ i < j \tag{9}$$

$$\sum_{j \in N(i)} x_{ij}^l \leq 2, \ n_i \in N, \ l \in L \tag{10}$$

$$h_l + \sum_{(n_i,n_j)\in A',\ i<j} x_{ij}^l = \sum_{n_i \in N} y_i^l,\ \ l \in L \tag{11}$$

$$\frac{1}{2} - \sum_{(n_i,n_j)\in A',\ i<j} x_{ij}^l + M(h_l - 1) \leq 0,\ \ l \in L \tag{12}$$

$$\frac{1}{2} - \sum_{(n_i,n_j)\in A',\ i<j} x_{ij}^l + M h_l \geq 0,\ \ l \in L \tag{13}$$

$$\sum_{n_i \in B} \sum_{n_j \in B} x_{ij}^l \leq |B| - 1, B \subseteq N,\ \ |B| \geq 2,\ \ l \in L \tag{14}$$

- Routing demand conservation constraints

$$\sum_{(n_i,n_p)\in A'} f_{ip}^w(s,t) + \sum_{(n_i,n_p)\in A''} \varphi_{ip}^w(s,t) = 0,\ \ w = (p,q) \in W \tag{15}$$

$$\sum_{(n_p,n_j)\in A'} f_{pj}^w(s,t) + \sum_{(n_p,n_j)\in A''} \varphi_{pj}^w(s,t) = 1,\ \ w = (p,q) \in W \tag{16}$$

$$\sum_{(n_i,n_q)\in A'} f_{iq}^w(s,t) + \sum_{(n_i,n_q)\in A''} \varphi_{iq}^w(s,t) = 1,\ \ w = (p,q) \in W \tag{17}$$

$$\sum_{(n_q,n_j)\in A'} f_{qj}^w(s,t) + \sum_{(n_q,n_j)\in A''} \varphi_{qj}^w(s,t) = 0,\ \ w = (p,q) \in W \tag{18}$$

$$\sum_{(n_i,n_k)\in A'} f_{ik}^w(s,t) - \sum_{(n_k,n_j)\in A'} f_{kj}^w(s,t) = 0,\ \ k \notin \{p,q\},\ \ w = (p,q) \in W \tag{19}$$

$$\sum_{(n_i,n_k)\in A''} \varphi_{ik}^w(s,t) - \sum_{(n_k,n_j)\in A''} \varphi_{kj}^w(s,t) = 0,\ \ k \notin \{p,q\},\ \ w = (p,q) \in W \tag{20}$$

$$f_{ij}^w(s,t) + \varphi_{ij}^w(s,t) \leq 1,\ \ (n_i,n_j) \in A,\ \ w \in W \tag{21}$$

- Location-Allocation constraints

$$f_{ij}^w(s,t) + p_w(s,t) - 1 \leq \sum_{l \in L} x_{ij}^l,\ \ w \in W \tag{22}$$

- Splitting demand constraints

$$\varepsilon + u_w^{RTN}(s,t) - u_w^{COM} - M\,(1 - p_w(s,t)) \leq 0,\ \ w \in W \tag{23}$$

$$u_w^{RTN}(s,t) = \Big(\sum_{(n_i,n_j)\neq(n_s,n_t)} d_{ij} f_{ij}^w(s,t)\Big) + (u_{st}^{COM} + \alpha_{st}) f_{st}^w(s,t) + \sum_{(n_i,n_j)\in A'} u_{ij}^{COM} \varphi_{ij}^w(s,t),\ \ w \in W \tag{24}$$

where u_{ij}^{COM} is the cost of traversing arc (n_i, n_j) using the complementary mode.

- Binary conditions

$$x_{ij}^l,\ x_{ij},\ y_i^l,\ h_l,\ f_{ij}^w(s,t),\ \varphi_{ij}^w(s,t), p_w(s,t) \in \{0,1\}.$$

where M is a sufficiently large real number and $\varepsilon > 0$ is a small tolerance.

Constraint (4) states that construction costs cannot exceed the budget, $C_{\max}$. Constraints (5) and (6) ensure that an arc is included in the RTN only if incident nodes are also selected. Constraints (7) allow the constructed arcs to be used in both directions. Constraints (8) and (9) impose that arc (n_i, n_j) is to be built if and only if there is a line that uses it. Constraints (10) states that each node is incident to at most two associated edges of each line. Constraints (11) ensure that a line does not contain a cyclic subgraph. Note that these constraints together with (11) guarantee that all constructed lines will be paths. However, a line must have at least one edge, which is ensured by constraints (12). If a line l is not considered in the design then it does not have any edge, constraints (13). Constraints (14) avoid subtours within lines.

Constraints (15) to (21) are flow conservation constraints at each node. The first node of a line has an outflow of 1 and the last node has an inflow of -1. Constraints (22) guarantee that a demand pair can be routed on an arc only if such an arc belongs to the rapid transit network. Constraints (23) force demand pairs to be assigned to the rapid transit mode if the associated cost of using this network (taking the fastest route) does not exceed the corresponding cost of the complementary mode. Note the role of the constant ε in these constraints to break possible ties. Finally, constraint (24) is the cost of satisfying the demand w if arc (s,t) fails, where α_{st} is the extra time spent from going to station n_s to the departure point of the alternative mode plus going from the arrival point of the alternative mode to station n_t.

As opposed to the deterministic model presented in the appendix, the model just described explicitly considers arc failures. When an arc fails, the affected passengers have three options: they can be rerouted on the network, they can use the complementary mode on the affected arc, they can complete their journey with the complementary mode. The solution to the problem described by (3) is a robust optimal network in the maximin sense, taking the trip coverage as the utility function. This is so because (3) is the detailed mathematical formulation of (2) which is equivalent to (1).

Due to the high complexity of the problem defined by (3), it is necessary to design practical ways of solving it. One option is to reduce the number of feasible networks. This makes sense since the maximin optimal network may not have a competitive trip coverage when failures do not occur (which is the usual case). Therefore we will impose that the resulting network should have at least some proportion of the trip coverage corresponding to the optimal network. The idea is to repeatedly solve the RTN problem described in the appendix, because it is faster to solve than (3) due to its lower number of variables and constraints. Once we have built a set of "good" networks, we choose the one maximizing the minimum utility when a link fails. These ideas are summarized in the following pseudocode:

- Let r^* be an optimal network solution for the RTN problem, and set $\alpha = \min_{a \in A'} K(r^*, a)$, $r^{SOL} = r^*$, $\sigma \in (0,1)$ and $\widetilde{R} = \{r^*\}$.
- Until break

  Calculate the optimal network $r \in R \setminus \widetilde{R}$.

  If $K(r) \geq \sigma K(r^*)$ then

$\widetilde{R} = \widetilde{R} \cup \{r\}$

if $\min_{a \in A'} K(r,a) > \alpha$ then

$r^{SOL} = r$ and $\alpha = \min_{a \in A'} K(r,a)$

end if.

Else

break

end if.

- **return** r^{SOL}

Note that $\sigma K(r^*)$ yields the minimum trip coverage aimed for in our set of feasible networks.

3 Robust Network Design as a Game

The main novelty of this paper is the formulation of the RND problem posed in Section 2 as a zero-sum two-player non-cooperative game, which is done in this section.

Borrowing from Aumann and Hart (1999), "Game Theory studies the behaviour of decision-makers (*players*) whose decisions affect each other". When studying games, one may assume that either all players will cooperate with each other or that the game will be played non-cooperatively. In a non-coooperative game a group of players converge and act, without the possibility of making agreements, in order to obtain certain benefit (*payoff*) by selecting one of their available strategies, given that other players will react by applying other strategies. When every player knows the possible actions that other players can take and the corresponding payoffs that would result, the game is said to have perfect information. Games are also divided according to the number of players involved in them. An introduction to game theory can be found in Forgo (1999).

In this paper we will model RND problem as a non-cooperative two-person zero-sum game, meaning that the game consists of two players each having its set of strategies. The players cannot make agreements and, what one player gains is lost by the opponent. In our case, player I is the RTN designer and operator and has as many (pure) strategies $r \in \mathcal{R}$ as the number of networks that can be built. Player II, a demon, has as many (pure) strategies $a \in A'$ as the number of links that can fail.

This kind of games can be simply represented by a matrix $A = (a_{ij})$ representing the payoff obtained by player I if he chooses his i^{th} strategy and player II chooses his j^{th} strategy. Player II then obtains a payoff equal to $-a_{ij}$. In the RND problem player I wants to maximize a certain utility function by building an RTN, while player II aims at minimizing it, by making a link fail. Therefore, the payoff of player I is the utility function (for instance the trip coverage) of the network built if the arc chosen by player II is inoperative, whereas the payoff of player II is the opposite. If we denote by $K(r,a)$ the utility of the network $r \in \mathcal{R}$ when arc $a \in \mathcal{A}'$

fails, the payoff matrix of this game has the form

$$\left(\begin{array}{cccc} K(r_1,a_1) & K(r_1,a_2) & ... & K(r_1,a_{|A'|}) \\ K(r_2,a_1) & K(r_2,a_2) & ... & K(r_2,a_{|A'|}) \\ ... & ... & ... & ... \\ K(r_{|R|},a_1) & K(r_{|R|},a_2) & ... & K(r_{|R|},a_{|A'|}) \end{array}\right).$$

A pair of strategies, (r^*, a^*) is called a *saddle point* if $K(r, a^*) \leq K(r^*, a^*) \leq K(r^*, a)$, $\forall\ r \in R$, $\forall\ a \in A^{'}$. Thus, $K(r^*, a^*)$ is the guaranteed trip coverage for player I against any arc player II chooses. It is also the maximum damage that player II can inflict to any network player I chooses. If the game has a saddle-point, then it satisfies

$$K(r^*, a^*) = \max_{r\in R}\min_{a\in A'} K(r, a) = \min_{a\in A'}\max_{r\in R} K(r, a), \tag{25}$$

and the strategy (r^*, a^*) is a so-called Nash equilibrium strategy, which means that no player can benefit by changing its strategy unilaterally.

If no saddle point exists it is possible for players to enlarge the available set of strategies, by considering probability vectors, and look for a saddle point in the enlarged game. In the enlarged game players can choose a convex combination of their pure strategies, having this way a *mixed strategy.* The mixed strategy $r_\beta = r_{(\beta_1,\ldots,\beta_{|R|})}$ for player I means that it would build network r_i with probability β_i, for every $i = 1, \ldots, |R|$. Analogously, the mixed strategy $a_\gamma = a_{(\gamma_1,\ldots,\gamma_{|A|})}$ of player II means that it will attack link a_i with probability γ_i. For the sake of readability, we will consider that the strategy (β, γ) is such that player I chooses the mixed strategy r_β and player II chooses the mixed strategy a_γ, for all $\beta \in \mathbb{R}_+^{|R|} \cup \{0\}$, $\gamma \in \mathbb{R}_+^{|A'|} \cup \{0\}$ satisfying $\sum_{i=1}^{|R|} \beta_i = \sum_{j=1}^{|A'|} \gamma_j = 1$. Note that the payoff for player I associated to the strategy (β, γ) is $K(\beta, \gamma) = \sum_{i=1}^{|R|}\sum_{j=1}^{|A'|} K(r_i, a_j)\beta_i\gamma_j$.

Definition 2 *A saddle point (β^*, γ^*) of the enlarged game is a pair of mixed strategies satisfying that, if β and γ are any mixed strategies for players I and II, then the expected payoff $K(\beta, \gamma)$ for player I satisfies*

$$K(\beta, \gamma^*) \leq K(\beta^*, \gamma^*) \leq K(\beta^*, \gamma).$$

The amount $K(\beta^*, \gamma^*)$ is called the value of the game. The pair (β^*, γ^*) are also called optimal mixed strategies.

The Von Neumann minimax theorem (1928), ensures that there always exists such a saddle point in the enlarged game. Note that the saddle point of the enlarged game leads to a better expected payoff for player I than the strategy obtained by solving the $\max_r \min_a K(r, a)$ problem, as stated in the following theorem.

Theorem 1 *Let β^*, γ^* be a pair of optimal mixed strategies for player I and player II, and r^m, a^m be an optimal robust network in the maximin sense and the arc that gives the minimum utility for that network when it fails, respectively. Then*

$$\sum_{i=1}^{|R|}\sum_{j=1}^{|A'|} \beta_i^*\gamma_j^* K(r_i, a_j) \geq K(r^m, a^m).$$

In other words, the previous theorem says that, in expected value, the optimal mixed strategy gives player I a higher utility than the optimal robust network in the maximin sense. Note that strategy $\beta = (\beta_1, \ldots, \beta_{|R|})$ states that we build one and only one of the possible networks. Which network should be built is decided from the corresponding probability vector β. These concepts are illustrated in the next section via two examples.

4 Examples

We consider the same network as in Laporte et al. (2008) (see Figure 1). There are nine nodes (potential stations) and 13 possible links, the latter forming the set A'. Each node has an associated construction cost c_i and each edge a pair (c_{ij}, d_{ij}),

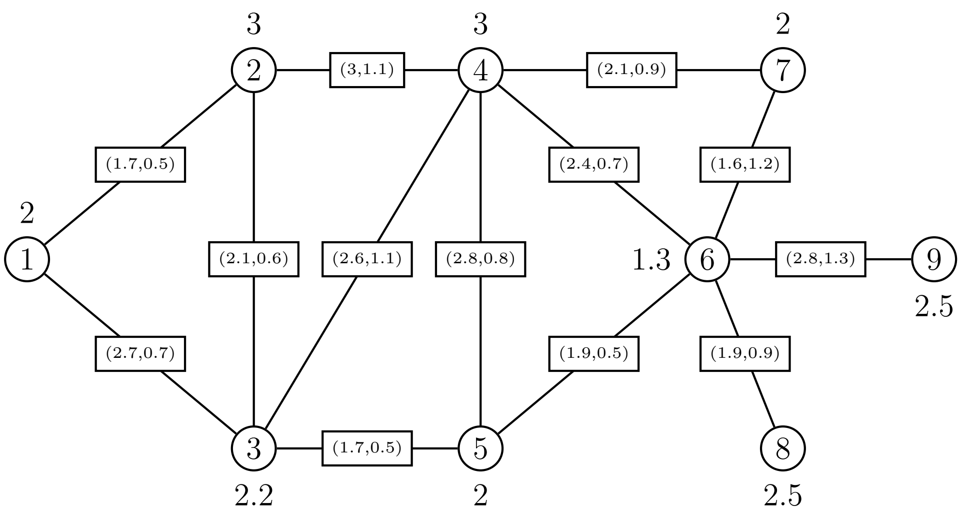

Figure 1: Test network.

c_{ij} and d_{ij} being the construction cost of the edge and its length, respectively (the latter parameter can also represent the generalized cost of traversing the edge by the rapid transit system).

The origin-destination demands g_w and their travel times via the alternative mode u_w^{COM} for each demand pair $w \in W$ are given by the matrices G and U^{COM}:

$$G = \begin{pmatrix} 0 & 9 & 26 & 19 & 13 & 12 & 4 & 6 & 4 \\ 11 & 0 & 14 & 26 & 7 & 18 & 3 & 7 & 9 \\ 30 & 19 & 0 & 30 & 24 & 8 & 3 & 9 & 11 \\ 21 & 9 & 11 & 0 & 22 & 16 & 21 & 18 & 16 \\ 14 & 14 & 8 & 9 & 0 & 20 & 12 & 18 & 9 \\ 26 & 1 & 22 & 24 & 13 & 0 & 11 & 28 & 21 \\ 7 & 5 & 6 & 19 & 15 & 13 & 0 & 16 & 14 \\ 5 & 9 & 11 & 16 & 17 & 25 & 17 & 0 & 21 \\ 6 & 8 & 10 & 18 & 11 & 20 & 14 & 20 & 0 \end{pmatrix};$$

$$U^{COM} = \begin{pmatrix} 0 & 1.6 & 0.8 & 2 & 1.6 & 2.5 & 4 & 3.6 & 4.6 \\ 2 & 0 & 0.9 & 1.2 & 1.5 & 2.5 & 3.2 & 3.5 & 4.5 \\ 1.5 & 1.4 & 0 & 1.3 & 0.9 & 2 & 3.3 & 2.9 & 3.9 \\ 1.9 & 2 & 1.9 & 0 & 1.8 & 2 & 2 & 3.8 & 4.1 \\ 3 & 1.5 & 2 & 2 & 0 & 1.5 & 3 & 2 & 3 \\ 2.1 & 2.7 & 2.2 & 1 & 1.5 & 0 & 2.5 & 3 & 2.5 \\ 3.9 & 3.9 & 3.9 & 2 & 3 & 2.5 & 0 & 2.5 & 2.5 \\ 5 & 3.5 & 4 & 4 & 2 & 3 & 2.5 & 0 & 2.5 \\ 4.6 & 4.5 & 4 & 3.5 & 3 & 2.5 & 2.5 & 2.5 & 0 \end{pmatrix}.$$

We will use the game explained in Section 3 to find a robust network, considering that the available budget is 40 units. Two utility functions will be considered: trip coverage and total travel time.

4.1 Trip coverage

We first consider as a primary objective the trip coverage of the network. Therefore, we aim at finding the network r^* and the arc a^* satisfying:

$$TC(r^*, a^*) = \max_{r \in R} \min_{a \in A'} TC(r, a), \tag{26}$$

$TC(r, a)$ being the trip coverage of network $r \in R$ when arc $a \in A'$ fails.

We will consider that the strategies for player I are all those networks that have at least some 95% of the trip coverage of the optimal network without failures. There are two reasons for this:

- for the sake of readability: the number of feasible networks is way too large to be represented in this paper;
- for operational purposes: the network solving problem in Equation (26) may have a low trip coverage when no arc fails; therefore we impose that our optimal robust network should have at least a certain trip coverage.

In this example the optimal network r_1 has a trip coverage of 831. The first six networks in terms of trip coverage are shown in Table 1. Therefore, the possible strategies for player I are the first five networks, because they have a trip coverage larger than or equal to $831 \times 0.95 = 789.45$. Table 2 gives the five networks satisfying this condition, and the trip coverage obtained in case any of the possible edges fails. Note that arcs $(2, 4)$ and $(4, 5)$ are not shown because they are not part of any of

Network	Lines	Trip coverage
r_1	(1-2-3-5-6-7), (4,6,9), (6,8)	831
r_2	(2-1-3-5-6-8), (7-4-6-9)	825
r_3	(1-2-3-5-6-8), (7-4-6-9)	795
r_4	(1-3-4-7-6-8), (3-5-6-9)	792
r_5	(1-3-4-6-8), (2-3-5-6-7)	791
r_6	(1-3-2), (4-3-5-6-8), (6-9)	783

Table 1: Optimal networks in terms of trip coverage.

	(1,2)	(1,3)	(2,3)	(3,4)	(3,5)	(4,6)	(4,7)	(5,6)	(6,7)	(6,8)	(6,9)
r_1	723	831	629	831	569	657	831	490	674	588	647
r_2	729	596	825	825	548	615	709	461	825	615	641
r_3	687	795	596	795	536	585	679	457	795	585	611
r_4	792	599	792	680	577	792	759	579	735	565	625
r_5	791	588	665	712	670	711	791	655	639	589	791

Table 2: Trip coverage optimal networks when edges fail.

the considered networks, and therefore their failures do not affect the trip coverage.

One can see that in this case the maximin strategy is $(r_5,(1,3))$, with a value of 588, whereas the minimax strategy is $(r_2,(6,8))$, with a value of 615. Since these two values do not coincide there is no saddle point strategy for this game. Therefore the most we can say is that if player I wants to maximize its worst case scenario, then network r_5 must be built. The minimax strategy states that the highest damage player II can guarantee is to decrease the trip coverage to, at least, 615 by attacking arc $(6,8)$. However, a better expected payoff is possible for player I by considering the set of enlarged strategies and by calculating a saddle point for this enlarged game.

In our example, we have to find the pair of probability vectors $\beta^* = (\beta_1^*, \ldots, \beta_5^*)$ and $\gamma^* = (\gamma_1^*, \cdots, \gamma_{11}^*)$ such that for a unique constant v the following conditions hold

$$\begin{array}{l} \sum_{i=1}^{5} TC(r_i,a_j)\beta_i^* \geq v, \quad j = 1,\ldots,11. \\ \sum_{j=1}^{11} TC(r_i,a_j)\gamma_j^* \leq v, \quad i = 1,\ldots,5. \end{array}$$

In our example

$$\beta_1^* = 0.025,\ \beta_2^* = 0.281,\ \beta_3^* = 0.694,\ \gamma_2^* = 0.079,\ \gamma_8^* = 0.112,\ \gamma_{10}^* = 0.809,$$

the other variables being equal to zero, and $v = 596.293$. This means that, by building network r_1 with probability 0.025, r_2 with probability 0.281 and r_5 with probability 0.694, player I ensures an expected trip coverage of 596.293, no matter which arc player II decides to attack, and this is the highest expected trip coverage he can guarantee. Such a trip coverage is achieved if player II decides to attack either edge $(1,3)$, $(5,6)$ or $(6,8)$ (edges 2, 8 and 10 in our ordering, respectively). Note that

Network	Lines	Total travel time
r_1	(1-2-3-5-6-7), (4,6,9), (6,8)	1746.8
r_2	(1-3-5-6-4-7), (7-6-8), (6-9)	1816
r_3	(2-1-3-5-6-4-7), (8-6-9)	1823
r_4	(2-1-3-5-6-7-4), (8-6-9)	1826.2
r_5	(1-2-3-5-6-4-7), (8-6-9)	1850.6

Table 3: Optimal networks in terms of total travel time.

	(1,2)	(1,3)	(2,3)	(3,5)	(4,6)	(4,7)	(5,6)	(6,7)	(6,8)	(6,9)
r_1	1914	1746	2085	2278	1966	1746	2458	1991	2487	2178
r_2	1816	2071	1816	2270	1961	1856	2475	1956	2504	2205
r_3	1956	2181	1823	2372	2137	1965	2566	1823	2446	2248
r_4	1961	2191	1826	2383	1826	1941	2568	2138	2514	2203

Table 4: Travelling times when arc failures occur.

if player II decides to attack any other arc, the expected trip coverage increases. This is why the strategy (β^*, γ^*) is a Nash Equilibrium in the enlarged game. Therefore, to maximize the minimum expected trip coverage when one arc fails, one must build either network r_1 or network r_2 or network r_5, with the probabilities used in this example.

4.2 Total travel time

We now consider as a primary objective the total travel time of the whole system (the network to be constructed plus those passengers using alternative transportation modes). Therefore, we seek to construct a network r^* satisfying:

$$TT(r^*, a^*) = \min_{r \in R} \max_{a \in A'} TT(r, a), \tag{27}$$

$TT(r, a)$ being the total travel time of the whole system when network $r \in R$ is constructed and $a \in A'$ fails.

For the same reasons as before we will consider that the strategies for player I are all those networks for which the total travel time of the system is at most 105% of the total travel time when the optimal network is working without failures. In our example the optimal network in terms of total travel time is the same as the one we calculated when the objective was trip coverage, which will again be denoted by r_1. The first five networks in terms of travel time are shown in Table 3. Therefore, the possible strategies for player I are the first four networks, because they yield a total travel time less than or equal to $1746.8 \times 1.05 = 1844.64$. In Table 4 we see the four networks that satisfy such a condition, and the total travel time of the corresponding network when any of the possible arcs fails. Again edges $(2, 4)$, $(3, 4)$ and $(4, 5)$ are not shown because they are not part of any of the considered networks, and therefore their failures do not affect the total travel time.

One can see that in this case the minimax strategy is $(r_1, (6, 8))$. Therefore, by building network r_1 player I guarantees a total travel time of no more than 2487 units. Note that, unlike what happens in the trip coverage example, in this case the optimal network when the objective is to minimize the total travel time coincides with the robust network in the minimax sense. The maximin strategy is $(r_2, (1, 2))$, whereas the saddle point of the enlarged game results in the probability vectors (β^*, γ^*):

$$\beta_1^* = 0.805,\ \beta_3^* = 0.195,\ \gamma_7^* = 0.275, \gamma_9^* = 0.725,\ v = 2479.020,$$

their other components being null. This means that, in terms of total travel time, the best response to a possible attack in an arc is to build network r_1 with probability 0.805 and network r_3 with probability 0.195, which guarantees an expected total travel time of at most 2479.02 units, which is almost eight units lower than the total travel time guaranteed by building the minimax network r_1 with probability 1.

5 Conclusions

We have introduced the Robust Network Design (RND) problem, in which the aim is to build the Rapid Transit Network (RTN) maximizing its minimum utility when one arc fails (maximin network). We have shown that such a problem can be modelled as an ILP. The solution to this program is a network that maximizes the minimum utility when one edge is inoperative: this is called a maximin network. In order to increase its efficiency we reduce the number of feasible networks by bounding their utility. We have formulated the RND problem as a non-cooperative game between the company that builds and exploits the network and an evil entity that wants to make arcs fail. The game is non-cooperative because the company does not know which arc will fail, and the evil entity does not know which network will be built. When this game has a saddle point, it corresponds to the optimal robust solution. Otherwise an enlarged game in which strategies are chosen probabilistically always contains a saddle point strategy, where the company is unable to increase the utility and the evil entity is unable to decrease it. We have provided an example, in which two utilities are considered: trip coverage (to be maximized) and total travel time (to be minimized).

Appendix: the RTN design model

In the model for the RTN design problem it is assumed that the mobility patterns in a metropolitan area are known. This implies that the number of potential passengers from each origin to each destination is given. It is also assumed that the locations of potential stations are known. In addition, there already exists a different mode of transportation (for example a bus network) competing with the RTN. To decide which mode each demand is allocated to, the generalized costs of the travellers for both systems are compared. The aim of the model is to design a network (i.e. to decide at which nodes stations are to be located and which edges must be constructed) consisting of lines covering as many trips as possible. Since the resources are limited there is also a budget constraint on the construction costs. The model uses the same data and notation described in Section 2. The variables used are also similar to the model introduced in the paper: y_i^l, x_{ij}^l, x_{ij} and h_l as before plus the following variables:

- $\widetilde{f}_{ij}^w$ denotes the normalized flow of the demand $w \in W$ through arc $(n_i, n_j) \in A'$, from n_i to n_j, $\widetilde{f}_{ij}^w \in \{0,1\}$ if no failure occurs. Note that such variables will define the fastest route for the demand w in the network to be built.
- $\widetilde{\varphi}_{ij}^w$ denotes the normalized flow of the demand $w \in W$ through arc $(n_i, n_j) \in A''$, from n_i to n_j, $\widetilde{\varphi}_{ij}^w \in \{0,1\}$ if no failure occurs.
- $\widetilde{p}_w = 1$, if demand w is allocated to the rapid transit network, that is, if its fastest route in the network takes less time than the alternative mode; 0, otherwise.

The objective of our model is to maximize the rapid transit trip coverage in the absence of failures

$$z_1 = \sum_{w=(p,q)\in W} g_w \widetilde{p}_w.$$

This model can be used to optimize other utility functions, such as the total travel time, by slightly changing it.

The constraints have been grouped according to their aims, where the budget constraints and the alignment location constraints are as in Section 2. The others are:

- Routing demand conservation constraints

$$\sum_{(n_i,n_p)\in A'} \widetilde{f}_{ip}^w + \sum_{(n_i,n_p)\in A''} \widetilde{\varphi}_{ip}^w = 0,\ w=(p,q)\in W$$

$$\sum_{(n_p,n_j)\in A'} \widetilde{f}_{pj}^w + \sum_{(n_p,n_j)\in A''} \widetilde{\varphi}_{pj}^w = 1,\ w=(p,q)\in W$$

$$\sum_{(n_i,n_q)\in A'} \widetilde{f}_{iq}^w + \sum_{(n_i,n_q)\in A''} \widetilde{\varphi}_{iq}^w = 1,\ w=(p,q)\in W$$

$$\sum_{(n_q,n_j)\in A'} \widetilde{f}_{qj}^w + \sum_{(n_q,n_j)\in A''} \widetilde{\varphi}_{qj}^w = 0,\ w=(p,q)\in W$$

$$\sum_{(n_i,n_k)\in A'} \widetilde{f}_{ik}^w - \sum_{(n_k,n_j)\in A'} \widetilde{f}_{kj}^w = 0,\ \text{if } k\notin\{p,q\},\ w=(p,q)\in W$$

$$\sum_{(n_i,n_k)\in A''} \widetilde{\varphi}_{ik}^w - \sum_{(n_k,n_j)\in A''} \widetilde{\varphi}_{kj}^w = 0,\ \text{if } k\notin\{p,q\},\ w=(p,q)\in W$$

$$\widetilde{f}_{ij}^w + \widetilde{\varphi}_{ij}^w \le 1,\ (n_i,n_j)\in A,\ w\in W$$

- Location-Allocation constraints

$$\widetilde{f}_{ij}^w + p_w - 1 \le \sum_{l\in L} x_{ij}^l,\ (n_i,n_j)\in A',\ w\in W$$

- Splitting demand constraints

$$\varepsilon + u_w - \mu u_w^{COM} - M(1 - \widetilde{p}_w) \leq 0, \; w = (p,q) \in W$$

where $u_w = \sum_{(n_i,n_j)\in A'} d_{ij} \widetilde{f}_{ij}^w + \sum_{(n_i,n_j)\in A''} u_{ij}^{COM} \widetilde{\varphi}_{ij}^w$, is a large enough real number, $\varepsilon > 0$ is a small tolerance and u_{ij}^{COM} is the generalized cost of traversing arc (i,j) by the complementary mode.

- Binary constraints

$$x_{ij}^l, \; x_{ij}, \; y_i^l, \; h_l, \; \widetilde{f}_{ij}^w, \; \widetilde{\varphi}_{ij}^w, \widetilde{p}_w \in \{0,1\}.$$

The interpretation of these constraints is similar to the ones introduced in Section 2. For a more detailed description of this model see Laporte et al. (2008).

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